Equations and inequalities 3B

1 a Rearrange $x + y = 11$ to give:
$y = 11 - x$
Substitute into $xy = 30$:
$x(11 - x) = 30$
$11x - x^2 = 30$
$0 = x^2 - 11x + 30$
$0 = (x - 5)(x - 6)$
$x = 5$ or $x = 6$
Substitute into $y = 11 - x$:
When $x = 5$, $y = 11 - 5 = 6$
When $x = 6$, $y = 11 - 6 = 5$
Solutions are $x = 5$, $y = 6$ or $x = 6$, $y = 5$

b Rearrange $2x + y = 1$ to give:
$y = 1 - 2x$
Substitute into $x^2 + y^2 = 1$:
$x^2 + (1 - 2x)^2 = 1$
$x^2 + 1 - 4x + 4x^2 = 1$
$5x^2 - 4x = 0$
$x(5x - 4) = 0$
$x = 0$ or $x = \frac{4}{5}$
Substitute into $y = 1 - 2x$:
When $x = 0$, $y = 1$
When $x = \frac{4}{5}$, $y = 1 - \frac{8}{5} = -\frac{3}{5}$
Solutions are
$x = 0$, $y = 1$ or $x = \frac{4}{5}$, $y = -\frac{3}{5}$

c $y = 3x$
Substitute into $2y^2 - xy = 15$:
$2(3x)^2 - x(3x) = 15$
$18x^2 - 3x^2 = 15$
$15x^2 = 15$
$x^2 = 1$
$x = -1$ or $x = 1$
Substitute into $y = 3x$:
When $x = -1$, $y = -3$
When $x = 1$, $y = 3$
Solutions are
$x = -1$, $y = -3$ or $x = 1$, $y = 3$

d Rearrange $3a + b = 8$ to give:
$b = 8 - 3a$
Substitute into $3a^2 + b^2 = 28$:
$3a^2 + (8 - 3a)^2 = 28$
$3a^2 + 64 - 48a + 9a^2 = 28$
$12a^2 - 48a + 36 = 0$
Divide by 12:
$a^2 - 4a + 3 = 0$
$(a - 1)(a - 3) = 0$
$a = 1$ or $a = 3$
Substitute into $b = 8 - 3a$:
When $a = 1$, $b = 8 - 3 = 5$
When $a = 3$, $b = 8 - 9 = -1$
Solutions are
$a = 1$, $b = 5$ or $a = 3$, $b = -1$.

e Rearrange $2u + v = 7$ to give:
$v = 7 - 2u$
Substitute into $uv = 6$:
$u(7 - 2u) = 6$
$7u - 2u^2 = 6$
$0 = 2u^2 - 7u + 6$
$0 = (2u - 3)(u - 2)$
$u = \frac{3}{2}$ or $u = 2$
Substitute into $v = 7 - 2u$:
When $u = \frac{3}{2}$, $v = 7 - 3 = 4$
When $u = 2$, $v = 7 - 4 = 3$
Solutions are
$u = \frac{3}{2}$, $v = 4$ or $u = 2$, $v = 3$

f Rearrange $3x + 2y = 7$ to give:
$2y = 7 - 3x$
$y = \frac{1}{2}(7 - 3x)$
Substitute into $x^2 + y = 8$:
$x^2 + \frac{1}{2}(7 - 3x) = 8$
Multiply by 2:
$2x^2 + (7 - 3x) = 16$
$2x^2 - 3x - 9 = 0$
$(2x + 3)(x - 3) = 0$
$x = -\frac{3}{2}$ or $x = 3$

1 f Substitute into $y = \frac{1}{2}(7 - 3x)$:

When $x = -\frac{3}{2}$, $y = \frac{1}{2}\left(7 + \frac{9}{2}\right) = \frac{23}{4}$

When $x = 3$, $y = \frac{1}{2}(7 - 9) = -1$

Solutions are

$x = -1\frac{1}{2}$, $y = 5\frac{3}{4}$ or $x = 3$, $y = -1$

2 a Rearrange $2x + 2y = 7$ to give:

$2x = 7 - 2y$

$x = \frac{1}{2}(7 - 2y)$

Substitute into $x^2 - 4y^2 = 8$:

$\left(\frac{1}{2}(7 - 2y)\right)^2 - 4y^2 = 8$

$\frac{1}{4}(7 - 2y)^2 - 4y^2 = 8$

Multiply by 4:

$(7 - 2y)^2 - 16y^2 = 32$

$49 - 28y + 4y^2 - 16y^2 = 32$

$0 = 12y^2 + 28y - 17$

$0 = (6y + 17)(2y - 1)$

$y = -\frac{17}{6}$ or $y = \frac{1}{2}$

Substitute into $x = \frac{1}{2}(7 - 2y)$:

When $y = -\frac{17}{6}$, $x = \frac{1}{2}\left(7 + \frac{17}{3}\right) = \frac{19}{3}$

When $y = \frac{1}{2}$, $x = \frac{1}{2}(7 - 1) = 3$

Solutions are

$x = 6\frac{1}{3}$, $y = -2\frac{5}{6}$ or $x = 3$, $y = \frac{1}{2}$

b Rearrange $x + y = 9$ to give:

$x = 9 - y$

Substitute into $x^2 - 3xy + 2y^2 = 0$:

$(9 - y)^2 - 3y(9 - y) + 2y^2 = 0$

$81 - 18y + y^2 - 27y + 3y^2 + 2y^2 = 0$

$6y^2 - 45y + 81 = 0$

Divide by 3:

$2y^2 - 15y + 27 = 0$

$(2y - 9)(y - 3) = 0$

$y = \frac{9}{2}$ or $y = 3$

Substitute into $x = 9 - y$:

When $y = \frac{9}{2}$, $x = 9 - \frac{9}{2} = \frac{9}{2}$

When $y = 3$, $x = 9 - 3 = 6$

Solutions are

$x = 4\frac{1}{2}$, $y = 4\frac{1}{2}$ or $x = 6$, $y = 3$

2 c Rearrange $5y - 4x = 1$ to give:

$5y = 4x + 1$

Substitute $y = \frac{4}{5}x + \frac{1}{5}$ into

$x^2 - y^2 + 5x = 41$:

$x^2 - (\frac{4}{5}x + \frac{1}{5})^2 + 5x = 41$

$x^2 - \frac{16}{25}x^2 - \frac{8}{25}x - \frac{1}{25} + 5x = 41$

$25x^2 - 16x^2 - 8x - 1 + 125x = 1025$

$9x^2 + 117x - 1026 = 0$

$x^2 + 13x - 114 = 0$

$(x + 19)(x - 6) = 0$

So $x = -19$ or $x = 6$

Substitute into $y = \frac{4}{5}x + \frac{1}{5}$

Solutions are $x = -19$, $y = -15$

or $x = 6$, $y = 5$

3 a Rearrange $x - y = 6$ to give:

$x = 6 + y$

Substitute into $xy = 4$:

$y(6 + y) = 4$

$6y + y^2 = 4$

$y^2 + 6y - 4 = 0$

Use the quadratic formula.

$a = 1, b = 6, c = -4$

$$y = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$= \frac{-6 \pm \sqrt{36 + 16}}{2}$$

$$= \frac{-6 \pm \sqrt{52}}{2}$$

$$= \frac{-6 \pm \sqrt{4 \times 13}}{2}$$

$$= -3 \pm \sqrt{13}$$

Substitute into $x = 6 + y$:

When $y = -3 - \sqrt{13}$,

$x = 6 - 3 - \sqrt{13} = 3 - \sqrt{13}$

When $y = -3 + \sqrt{13}$,

$x = 6 - 3 + \sqrt{13} = 3 + \sqrt{13}$

Solutions are

$x = 3 - \sqrt{13}$, $y = -3 - \sqrt{13}$

or $x = 3 + \sqrt{13}$, $y = -3 + \sqrt{13}$

3 b Rearrange $2x + 3y = 13$ to give:

$2x = 13 - 3y$

$x = \frac{1}{2}(13 - 3y)$

Substitute into $x^2 + y^2 = 78$:

$\left(\frac{1}{2}(13 - 3y)\right)^2 + y^2 = 78$

$\frac{1}{4}(13 - 3y)^2 + y^2 = 78$

Multiply by 4:

$(13 - 3y)^2 + 4y^2 = 312$

$169 - 78y + 9y^2 + 4y^2 = 312$

$13y^2 - 78y - 143 = 0$

Divide by 13:

$y^2 - 6y - 11 = 0$

Use the quadratic formula.

$a = 1,\ b = -6,\ c = -11$

$$y = \frac{-b \pm \sqrt{(b^2 - 4ac)}}{2a}$$

$$= \frac{6 \pm \sqrt{(36 + 44)}}{2}$$

$$= \frac{6 \pm \sqrt{80}}{2}$$

$$= \frac{6 \pm \sqrt{16 \times 5}}{2}$$

$$= \frac{6 \pm 4\sqrt{5}}{2}$$

$$= 3 \pm 2\sqrt{5}$$

Substitute into $x = \frac{1}{2}(13 - 3y)$:

When $y = 3 - 2\sqrt{5}$,

$$x = \tfrac{1}{2}\left(13 - 3\left(3 - 2\sqrt{5}\right)\right)$$

$$= \tfrac{1}{2}\left(13 - 9 + 6\sqrt{5}\right)$$

$$= 2 + 3\sqrt{5}$$

When $y = 3 + 2\sqrt{5}$,

$$x = \tfrac{1}{2}\left(13 - 3\left(3 + 2\sqrt{5}\right)\right)$$

$$= \tfrac{1}{2}\left(13 - 9 - 6\sqrt{5}\right)$$

$$= 2 - 3\sqrt{5}$$

Solutions are

$x = 2 - 3\sqrt{5},\ y = 3 + 2\sqrt{5}$

or $x = 2 + 3\sqrt{5},\ y = 3 - 2\sqrt{5}$

4 Rearrange $x + y = 3$ to give:

$y = 3 - x$

Substitute into $x^2 - 3y = 1$.

$x^2 - 3(3 - x) = 1$

$x^2 + 3x - 10 = 0$

$(x + 5)(x - 2) = 0$

So $x = -5$ or $x = 2$

Solutions are $x = -5$, $y = 8$ or $x = 2$ and $y = 1$

5 a $3x^2 + x(2 - 4x) + 11 = 0$

$3x^2 + 2x - 4x^2 + 11 = 0$

$-x^2 + 2x + 11 = 0$

$x^2 - 2x - 11 = 0$

b Using the quadratic formula,

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$x = \frac{2 \pm \sqrt{4 + 44}}{2}$$

$$x = \frac{2 \pm \sqrt{48}}{2}$$

$$x = \frac{2 \pm \sqrt{16 \times 3}}{2}$$

$$x = \frac{2 \pm 4\sqrt{3}}{2}$$

$x = 1 \pm 2\sqrt{3}$

Substitute into $y = 2 - 4x$:

$x = 1 + 2\sqrt{3},\ y = -2 - 8\sqrt{3}$

or $x = 1 - 2\sqrt{3},\ y = -2 + 8\sqrt{3}$

6 a At the point $(1, p)$, $x = 1$ and $y = p$.

Substituting these values into the first equation gives:

$p = k - 5$ **(1)**

Substituting these values into the second equation gives:

$4 - p = 6$ **(2)**

$p = -2$

When $p = -2$, $k = 3$

$k = 3$, $p = -2$

b Substitute for k into $y = kx - 5$:

$y = 3x - 5$

Substitute into $4x^2 - xy = 6$:

$4x^2 - x(3x - 5) = 6$

$4x^2 - 3x^2 + 5x - 6 = 0$

$x^2 + 5x - 6 = 0$

$(x - 1)(x + 6) = 0$

6 b $x = 1$ or $x = -6$
When $x = -6$, $y = -23$ and $x = 1$, $y = -2$
The solutions are $x = -6$, $y = -23$
or $x = 1$, $y = -2$

Challenge

Rearrange $y - x = k$ into $x^2 + y^2 = 4$:
$y = x + k$
$x^2 + (x + k)^2 = 4$
$x^2 + x^2 + 2kx + k^2 - 4 = 0$
$2x^2 + 2kx + k^2 - 4 = 0$
Using the discriminant for one pair of solutions,
$b^2 - 4ac = 0$
$(2k)^2 - 4(2)(k^2 - 4) = 0$
$4k^2 - 8k^2 + 32 = 0$
$-4k^2 = -32$
$k^2 = 8$
$k = \pm\sqrt{8}$
$= \pm\sqrt{4 \times 2}$
$= \pm 2\sqrt{2}$